

RIEMANN HYPOTHESIS MAY BE PROVED BY INDUCTION

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ABSTRACT. The transformations of the sum identities for generalized harmonic and oscillatory numbers, obtained earlier in our recent report [1], enable us to derive the new identities expressed in terms of the corresponding square roots of x . At least one of these identities may be applied to prove the Riemann Hypothesis by induction. Additionally using this approach, the new series for Euler's constant γ has been found.

Keywords: generalized harmonic number, generalized oscillatory number, distribution of primes, Riemann Hypothesis, proof by induction, Euler's constant γ

Let us start discussion from the sum identity

$$(1) \quad \sum_{i=1}^x \frac{1}{i^s} M_{\frac{x}{i}}(s) = 1$$

obtained in our recent work [1] for generalized oscillatory numbers in power s

$$M_x(s) = \sum_{k=1}^x \frac{\mu_k}{k^s},$$

where μ_k is the Möbius function. Rearranging $M_{\frac{x}{i}}(s)$ in terms of $\frac{(x^2/j)}{i}$, where $j \geq x$ is an integer variable, the equation (1) can be rewritten as

$$(2) \quad \sum_{i=1}^x \frac{1}{i^s} M_{\frac{(x^2/j)}{i}}(s) = 1, \quad j \geq x.$$

Multiplying both parts of (1) by $\frac{1}{j^s}$ and taking the sum over index j from $x+1$ up to x^2 , the right hand side can be represented as a difference of the generalized harmonic numbers in power s [2] for x^2

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and x , i.e.:

$$(3) \quad \sum_{j=x+1}^{x^2} \frac{1}{j^s} \sum_{i=1}^x \frac{1}{i^s} M_{\frac{(x^2/j)}{i}}(s) = \sum_{j=x+1}^{x^2} \frac{1}{j^s} = H_{x^2}(s) - H_x(s).$$

On the other hand we can rewrite (3) as

$$(4) \quad \begin{aligned} \sum_{i=1}^x \frac{1}{i^s} \sum_{j=x+1}^{x^2} \frac{1}{j^s} M_{\frac{(x^2/i)}{j}}(s) &= \sum_{i=1}^x \frac{1}{i^s} \left(1 - \sum_{j=1}^x \frac{1}{j^s} M_{\frac{(x^2/i)}{j}}(s) \right) \\ &= H_x(s) - \sum_{i,j=1}^x \frac{1}{(i \cdot j)^s} M_{\frac{x^2}{i \cdot j}}(s). \end{aligned}$$

Combining (3) and (4), we get the identity

$$(5) \quad H_{x^2}(s) = 2H_x(s) - \sum_{i,j=1}^x \frac{1}{(i \cdot j)^s} M_{\frac{x^2}{i \cdot j}}(s).$$

Consider two most interesting cases following from (5). At $s = 0$ $M_x(0) = \sum_{k=1}^x \mu_k \equiv M_x$ is Mertens function and we have

$$(6) \quad [x^2] = 2[x] - \sum_{i,j=1}^x M_{\frac{x^2}{i \cdot j}},$$

while at $s = 1$ $M_x(1) = \sum_{k=1}^x \frac{\mu_k}{k} \equiv m_x$, the formula (5) is an expression for harmonic number at x^2

$$(7) \quad H_{x^2} = 2H_x - \sum_{i,j=1}^x \frac{1}{i \cdot j} m_{\frac{x^2}{i \cdot j}}.$$

Applying now the asymptotic formula for harmonic number [2], we get

$$(8) \quad \log[x^2] + \gamma + O\left(\frac{1}{x^2}\right) = 2 \left(\log[x] + \gamma + O\left(\frac{1}{x}\right) \right) - \sum_{i,j=1}^x \frac{1}{i \cdot j} m_{\frac{x^2}{i \cdot j}},$$

where $\gamma = 0.5772156\dots$ is Euler's constant [3]. Hence immediately follows the new series for constant γ

$$(9) \quad \sum_{i,j=1}^x \frac{1}{i \cdot j} m_{\frac{x^2}{i \cdot j}} = \gamma + O\left(\frac{1}{x}\right)$$

or

$$(10) \quad \gamma = \lim_{x \rightarrow \infty} \sum_{i,j=1}^x \frac{1}{i \cdot j} m_{\frac{x^2}{i \cdot j}}.$$

By analogy with (1)-(7), we can obtain the similar set of equations for generalized harmonic numbers in power s $H_x(s)$. In particular, the corresponding counterparts (11)-(18), expressed in terms of generalized harmonic numbers, can be found:

$$(11) \quad \sum_{i=1}^x \frac{\mu_i}{i^s} H_{\frac{x}{i}}(s) = 1,$$

$$(12) \quad \sum_{i=1}^x \frac{\mu_i}{i^s} H_{\frac{(x^2/j)}{i}}(s) = 1, \quad j \geq x,$$

Multiplying both parts of (12) by $\frac{\mu_j}{j^s}$ and taking sum over index j from $x+1$ up to x^2 yields

$$(13) \quad \sum_{j=x+1}^{x^2} \frac{\mu_j}{j^s} \sum_{i=1}^x \frac{\mu_i}{i^s} H_{\frac{(x^2/j)}{i}}(s) = M_{x^2}(s) - M_x(s),$$

$$(14) \quad \begin{aligned} \sum_{i=1}^x \frac{\mu_i}{i^s} \sum_{j=x+1}^{x^2} \frac{\mu_j}{j^s} H_{\frac{(x^2/i)}{j}}(s) &= \sum_{i=1}^x \frac{\mu_i}{i^s} \left(1 - \sum_{j=1}^x \frac{\mu_j}{j^s} H_{\frac{(x^2/i)}{j}}(s) \right) \\ &= M_x(s) - \sum_{i,j=1}^x \frac{\mu_i \mu_j}{(i \cdot j)^s} H_{\frac{x^2}{i \cdot j}}(s). \end{aligned}$$

By analogy with (5) we get the identity

$$(15) \quad M_{x^2}(s) = 2M_x(s) - \sum_{i,j=1}^x \frac{\mu_i \mu_j}{(i \cdot j)^s} H_{\frac{x^2}{i \cdot j}}(s).$$

Consider again two most interesting cases. At $s = 1$ (15) gives

$$(16) \quad m_{x^2} = 2m_x - \sum_{i,j=1}^x \frac{\mu_i \mu_j}{i \cdot j} H_{\frac{x^2}{i \cdot j}},$$

while at $s = 0$ we have

$$(17) \quad M_{x^2} = 2M_x - \sum_{i,j=1}^x \mu_i \mu_j \left[\frac{x^2}{i \cdot j} \right]$$

or

$$(18) \quad M_{x^2} = 2M_x - x^2 \cdot m_x^2 + \sum_{i,j=1}^x \mu_i \mu_j \left\{ \frac{x^2}{i \cdot j} \right\}.$$

Let us consider identity (17) in more detail. Assume the Riemann Hypothesis. In this case for all $\varepsilon > 0$ $M_x = O\left(x^{\frac{1}{2}+\varepsilon}\right)$, which is equivalent to $|M_x| \leq C\sqrt{x}(\log x)^n$ for the given positive constants C and n . For x^2 this can be rewritten as $|M_{x^2}| \leq C \cdot 2^n \cdot x(\log x)^n$. Obviously, if $M_x = o(\sqrt{x} \log x)$ then some positive function $f(x)$ satisfying the conditions $f(x) = o(\log x)$ and $M_x = O(\sqrt{x}f(x))$ can be used instead of $\log x$.

Let us formulate *the Induction Procedure* for the reverse statement.
Induction Procedure

Assume that for some real bound $x_0 > e$, variables x and y , and for the given positive constants C and n

$$(19) \quad \sup |M_y| = \sup \left| \sum_i^y \mu_i \right| \leq C\sqrt{x}(\log x)^n, \quad e \leq y \leq x < x_0.$$

In first step the verification of this assumption can be done by direct calculation of $|M_k|$ for all $k < x_0$, where k is the natural number.

If applying the identity (17) we can prove that always (regardless of x_0 value)

$$(20) \quad \begin{aligned} \sup |M_{y^2}| &\sim \sup \left| \sum_{i,j=1}^y \mu_i \mu_j \left[\frac{y^2}{i \cdot j} \right] \right| \\ &\leq 2^n \cdot \sqrt{x} \cdot \sup |M_y|, \end{aligned} \quad e \leq y \leq x < x_0,$$

then we confirm that

$$\sup |M_y| \sim \sup \left| \sum_{i,j=1}^{\sqrt{y}} \mu_i \mu_j \left[\frac{y}{i \cdot j} \right] \right| \leq C\sqrt{x}(\log x)^n, \quad e \leq y \leq x < x_0^2,$$

and the statement (19) is extended now up to x_0^2 .

End of Induction Procedure

The Induction Procedure can be applied over and over again for further validation of (19). Hence the Riemann Hypothesis is justified.

Thus, to prove the Riemann Hypothesis it is enough to prove that:

if for some real bound x_0 , variables x and y , and for the given positive constants C and n

$$(21) \quad \sup \left| \sum_i^y \mu_i \right| \leq C\sqrt{x}(\log x)^n, \quad e \leq y \leq x < x_0,$$

then always (independently of x_0 value) follows

$$\begin{aligned}
 (22) \quad & \sup \left| \sum_{i,j=1}^y \mu_i \mu_j \left[\frac{y^2}{i \cdot j} \right] \right| = \sup \left| y^2 m_y^2 - \sum_{i,j=1}^y \mu_i \mu_j \left\{ \frac{y^2}{i \cdot j} \right\} \right| \\
 & \leq 2^n \cdot \sqrt{x} \cdot \sup \left| \sum_i^y \mu_i \right|, \quad e \leq y \leq x < x_0.
 \end{aligned}$$

REFERENCES

- [1] R. M. Abrarov, S. M. Abrarov, *On the properties of generalized harmonic and oscillatory numbers. Simple proof of the Prime Number Theorem*, http://arxiv.org/PS_cache/arxiv/pdf/0709/0709.3145v2.pdf
- [2] <http://mathworld.wolfram.com/HarmonicNumber.html>
- [3] <http://mathworld.wolfram.com/Euler-MascheroniConstant.html>

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